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A NOTE ON SMALE'S GLOBAL NEWTON METHOD

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We present an implementation of Smale's Global Newton method in a simple setting. The iteration count for the algorithm is sensitive only to the quantity β_0 defined below.

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A note on Smale's Global Newton Method

A. A. Goldstein*

We have previously considered Smale's [1] global Newton method in conjunction with the Kantorovich inequalities in a simple setting. Our object was to determine a class of problems for which the method could be expected to be efficient. We now repeat this project using Smale's [2] estimates at one point instead of the Kantorovich estimate.

The power of Smale's estimate stems not only from the fact that the information is concentrated at one point. A further advantage is that, in contrast with the Kantorovich inequalities, no estimate of the norm of the inverse of the derivative operator by itself is needed. Newton step lengths can be small even if the derivative operator is nearly singular.

The algorithm below requires only local information for its implementation. To predict its behaviour, however, certain global constants are needed. These constants can be estimated on a thin cylinder containing a segment joining the origin and the initial value of the vector valued function for which we are seeking the root. The iteration count for the algorithm is sensitive only to the parameter $\beta\gamma$ defined below.

Theorem 1 (Smale 86) Assume F is an analytic map between real Banach spaces X and Y. That is, the Frechet derivatives $F^{(k)}(x)$ exist for all x $\in$ X and k=1,2,3,.....Given $x_0 \in$ X, assume that the inverse of $F'(x)$, which we denote by $F'_{-1}(x)$, exists. Set

$$\beta(x_0) = ||F'_{-1}(x_0)F(x_0)|| \qquad \textit{and}$$

$$\gamma(x_0) = sup\left\{ ||\frac{1}{k!}F'_{-1}(x_0)F^{(k)}(x_0)||^{\frac{1}{k-1}} : k \geq 2 \right\}$$

If

$$\beta(x_0)\gamma(x_0) \; < \; .130707$$

then x_0 is an approximate root of F. That is the Newton sequence

$$x_{k+1} = x_k - F'_{-1}(x_k)F(x_k)$$

is well defined and $\{x_k\}$ converges to say ξ, a root of F at the rate:

$$||x_{k+1} - x_k|| \; \leq \; 2\left(\frac{1}{2}\right)^{2^k} \beta(x_0)$$

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Moreover,

$$\|x_k - \xi\| \;\le\; \frac{7}{4}\left(\frac{1}{2}\right)^{2^{k-1}} \beta(x_0)$$

The next result is not given by Smale but it is readily calculated using his ingredients. In what follows we shall often abbreviate $\beta(x_k)$ (and similar expressions) by β_k.

Remark 2

$$\|F'(x_k)\| \;<\; 1.85\,\|F'(x_0)\| \qquad k = 1, 2, 3, \dots\dots\dots$$

Proof. In [2], Proposition 2, we find the formula

$$\alpha_{k+1} \;\le\; \alpha_k \,/(2\alpha_k^2 \;-4\,\alpha_k \;+\; 1),$$

where $\alpha_k \;=\; \beta_k\,\gamma_k \;\le\; 1/8$. Then $\alpha_{k+1} \;<\; (2\,\alpha_k)^2$. In Smale's proof of Lemma 2 we find $\|\,I - F'_{-1}(x_k)F'(x_{k+1})\,\| \;\le\; (1-\;\alpha_k)^{-2} \;-\; 1$. Whence $\|\,I - F'_{-1}(x_k)F'(x_{k+1})\,\| \;\le\; 1 \;+\; (11/4)\,\alpha_k$. Let $H_k = F'_{-1}(x_k)F'(x_{k+1})$ then $F'(x_{k+1}) = F'(x_k)\,H_k$, and

$$\|F'(x_{k+1})\| \;<\; \|F'(x_0)\| \prod_{i=0}^{k}(1 \;+\; (11/4)\,(2\,\alpha_0)^{2^i})$$

We can improve this result by using sharper bounds for i=0 and for i=1, namely, .306 and .138, respectively. Then

$$\log\left(\prod_{i=2}^{\infty}(1 \;+\; (11/4)\,(2\,\alpha_0)^{2^i})\right) \;<\; (11/4)\int_1^{\infty}(1/4)^{2^x}\,dx \;<\; .172$$

Finally we have the estimate

$$\prod_{i=0}^{\infty}\|H_k\| \;<\; 1.85.$$

Assume the hypotheses of Theorem 1. Let

$$T = \{x \in X : \|F(x)\| \;\le\; \|F(x_0)\|\}.$$

We assume that $\gamma(x)$, $F(x)$ and $\beta(x)$ are defined and can be calculated throughout T. Assume the existence of numbers β, γ, K and σ that bound $\beta(x)$, $\gamma(x)$, $\|F'(x)\|$ and

$$\sigma(x) = \|F'_{-1}(x)\frac{F(x)}{\|F(x)\|}\|$$

on T. Let]x[denote the smallest integer $\geq$ x.

Algorithm 3 Given $x_i \in T$, we define x_{i+1} as follows. Set $t_i = 1-(8\gamma_i\beta_1)^{-1}$, where $\gamma_i = \gamma(x_i)$ and β_1 is defined similarly. Set $\xi_0^i = x_i$ and run the Newton sequence for G starting at ξ_0^i. Let k be the smallest integer satisfying

$$G(\xi_k^i) \leq \|F(x_i)\|/40\gamma_i\beta_i$$

Set $x_{i+1} = \xi_k^i$.

The algorithm can be run and terminated with a posteriori data, without any knowledge of the values of the constants β, γ, σ, *and* K. Moreover to verify a) and b) below we do not use these constants. However the global behaviour of the algorithm as described in c) and d) is given in terms of these constants. Let $K_i = 1.85\,\|F(x_i)\|$.

Claim 4

(a) x_{i+1} can be found in $S =]\log_2(\log_2(80\,K_i\,\gamma_i\,\beta_i^2\,/\,\|F(x_i)\|))[$ steps.

(b) $\|F(x_{i+1})\| \leq (1-(1/10\,\gamma_i\beta_i))\,\|F(x_i)\|$

(c) Let S $= \log_2(\log_2(80\,K\,\gamma\,\beta\,\sigma))$. Given $\epsilon > 0$, if k $\geq]10\,\gamma\beta S\,\log_2(1/\epsilon)[$ then $(\|F(x_k)\,/\|F(x_0)) \leq \epsilon$.

(d) If k $\geq]10\,\gamma\beta S\log(8\,\gamma\sigma)[$ then x_k is an approximate root of F.

Proof. We show first that x_{i+1} can be chosen as claimed. Let $G^i(x) = t_iF(x_i)$. Then x_i is an approximate root for G^i. Hence if $\xi_0 = x_i$ and $\xi_{k+1} = \xi_k - F'(\xi_k)G(\xi_k)$ then ξ_k converges to ξ^i, a root of G^i. We now prove (b)

We have that $G^i(x_i) - G^i(\xi^i) = G^i(x_i) = (F(x_i)\,/\,8\,\beta_i\gamma_i) = F(x_i) - F(\xi_i)$. Also, $F(x_i) - F(x_{i+1}) = F(x_i) - F(\xi^i) + G^i(\xi^i) - G^i(x_{i+1}) = (F(x_i)\,/\,8\,\beta_i\gamma_i) + G^i(\xi^i) - G^i(x_{i+1})$. Thus b) is true.

We now prove a) by counting the steps needed to ensure that
$\|G^i(\xi^i)\| \leq \|F(x_i)\|\,/40\,\gamma_i\beta_i = tmp$. Since $G^i(\xi_i) = F'(\xi_i)\,(\xi_{i+1}-\xi_i)$
we may choose k to ensure that $2\,K^i(\frac{1}{2})^{2^k}\,\beta(x_i) \leq tmp$

To prove c) replace β_i *and* γ_i by β *and* γ, respectively. If k satisfies
$k\,\log_2(1-(1\,/\,10\,\gamma\beta)) \leq \epsilon$ then $(\|F(x_k)\|\,/\,\|F(x_0)\|) \leq \epsilon$.
To go from k to k+1 requires no more than S steps.

We now turn to d). If $\|F(x)\| \leq \frac{1}{8\,\gamma\sigma}$ then x is an approximate root for F. Thus we replace $1\,/\,\epsilon$ in c) by $8\,\sigma\gamma$

Remarks 5

The method is sensitive only to the number $\gamma\beta$!

Note that from the above proof $F(\xi^i) = t_i\,F(x_i)$. Since x_{i+1} lies near ξ^i the global constants need be estimated only on a slender tube surrounding the segment joining the origin and $F(x_0)$.

Bibliography

[1] Steve Smale Algorithms for solving Equations, Proceedings of the International Congress of Mathematicians, Vol1, pp172,195. AMS,1986.

[2] Steve Smale Newton's method estimates from data at one point. The Merging of Disciplines... Springer-Verlag 1986 185-196.

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